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On cages with given degree sets

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Abstract

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We consider the problem of constructing minimal graphs of given girth having a particular degree set.

1. Introduction

A (v, g)-*cage* (or *cage*) *is a* v-regular (simple) graph of girth g on the minimum number of vertices. For $v = 2$ the cage is a cycle; for $g = 3$ the cage is K_{v+1}; and for $g = 4$ the cage is $K_{v,v}$. The most famous cage is of course the (3, 5)-cage or the Petersen graph, Fig. 1 (see [4]).

The exact order of cages is a difficult and long standing problem. A comprehensive article in this area is the survey paper of Wong [5]. Cages have been generalized in a number of ways such as prescribing the minimum 'odd and even girth'. Another such generalization is as follows: if $D = \{a_1, a_2, \ldots, a_k\}$ is a set of positive integers with $2 \leq a_1 < \cdots < a_k$ then $f(D; g) = f(a_1, a_2, \ldots, a_k; g)$ is the minimum order of a graph with degree set D and girth g. Thus for example the ordinary (v, g)-cage has order $f(v; g)$. Few values of $f(D; g)$ are known beyond the special cases (see Chartrand et al [2]):

$f(3, 4; 5) = 13$ and $f(3, 4; 6) = 18$;
$f(D; 3) = a_k + 1$;
$f(2, m; g) = 1/2(m(g-2) + 4$ for g even and
$f(2, m; g) = 1/2(m(g-1) + 2)$ for g odd; and
$f(r, s; 4) = r + s$.

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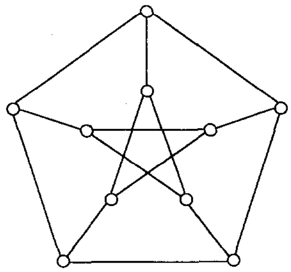


Fig. 1.

In [3], Downs et al determined $f(3, m; 5) = f(3, 4, m; 5) = 3m + 1$; $f(3, m; 7) = 7m + 1$; and $f(3, m; 9) = 15m + 1$. In this article we consider the problems of evaluating $f(3, m; 6)$ and $f(4, m; 5)$; in particular we conclude that $f(3, m; 6) = 4m + 2$ for $m \geq 3$ and that $f(4, m; 5) = 4m + 1$ for $m \geq 5$.

2. The $(3, m; 6)$-cages

To evaluate $f(3, m; 6)$ we will first show that the lower bound is $4m + 2$ and then use a construction to attain this bound. Suppose on the contrary that $f(3, m; 6) < 4m + 2$ and that G is a $(3, m; 6)$-cage. For convenience we choose a vertex x of degree m; let its neighbourhood set be denoted by $X = \{x_1, x_2, \ldots, x_m\}$; let the set of vertices at distance two from x be denoted by X'; finally denote the remaining vertices of G by Y where $|Y| = k$. We define two subgraphs, U and V, of a graph to be *d-remote* if the distance between them is at least d. We claim that any two points of degree m in G are 3-remote. Suppose for example that x_1 has degree m. Then girth 6 implies that the remaining neighbours of x_1 are not adjacent to the neighbours of $x_2, \ldots, x_m$ and we have $f(3, m; 6) \geq 1 + m + (m - 1) + (2(m - 1) + 2(m - 1) = 6m - 4 \geq 4m + 2$ whenever $m \geq 3$. Thus we may assume that $X' = \{x_{11}, x_{12}, x_{21}, x_{22}, \ldots, x_{m1}, x_{m2}\}$ and $|Y| \leq m$. Now suppose for example that x_{11} has degree m, then x_{11} has $m - 1$ neighbours in

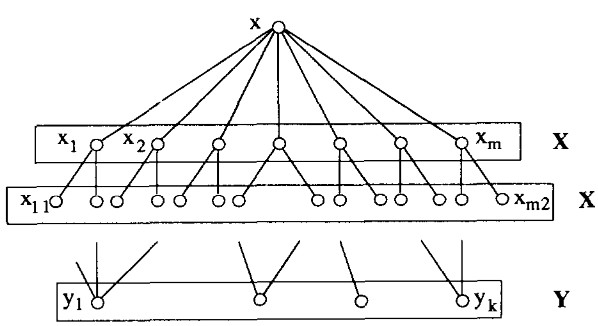
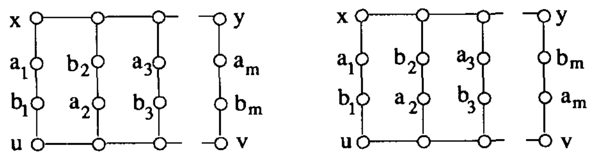


Fig. 2.



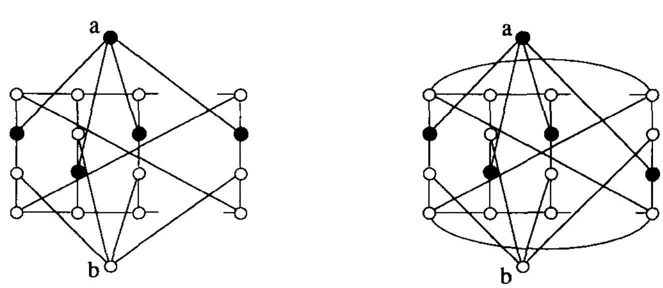
i) m odd ii) m even

Fig. 3.

Y and x_{12} has two further neighbours in Y, a contradiction. The 3-remoteness of such vertices says that G has the structure shown in Fig. 2 where all degrees in X and X' are equal to three.

If Y contains t points of degree m where $0 \leq t \leq 1$ then there are at most $tm + 3(k-t) \leq (3+t)m - 3t$ edges from Y to X', however this must be $4m$ using the girth 6 condition on the vertices of X', a contradiction. Suppose that $t \geq 3$; since these t vertices must be 3-remote we have that $|X' \cup Y| \geq 3m + 3$ and $f(3, m; 6) > 4m + 1$. Thus we must have precisely $t = 2$. Without loss of generality let y_1 and y_2 have degree m.

Let A be the subset of X consisting of those vertices at distance two from both y_1 and y_2; let B be the subset of X consisting of those vertices at distance two from exactly one of y_1 or y_2; let $C = X - (A \cup B)$. Let the neighbourhood set of A in X' be A'. Finally let Y_1 and Y_2 be the (disjoint) neighbourhood sets of y_1 and y_2 respectively in Y and let $Y_3 = Y - (Y_1 \cup Y_2 \cup \{y_1, y_2\})$. Note that no vertex y in $Y_1 \cup Y_2$ is at distance two from A else G would contain a 3-cycle or a 5-cycle. Similarly no vertex in Y_3 can have three neighbours in A' for this would force a 4-cycle with either y_1 or y_2. These two observations imply that each vertex in A' has at least one neighbour in Y_3, and that $|Y_3| \geq |A|$. Let E^+ denote the set of edges joining y_1 and y_2 to X'. Then $|E^+| = 2|A| + |B| \leq |A| + |X| = |A| + m$ leading to $|E^+| - m \leq |A|$. On the other hand we have that $|E^+| + |Y_1 \cup Y_2| = 2m$ while $|Y_1 \cup Y_2| + |Y_3| = k - 2 \leq m - 2$. In turn we have that $|Y_3| \leq (m-2) - |Y_1 \cup Y_2| = (m-2) - (2m - |E^+|) = |E^+| - m - 2$. However as we have shown earlier, $|Y_3| \geq |A|$, and we have a contradiction. It follows that $f(3, m; 6) \geq 4m + 2$.



i) m odd ii) m even

Fig. 4.

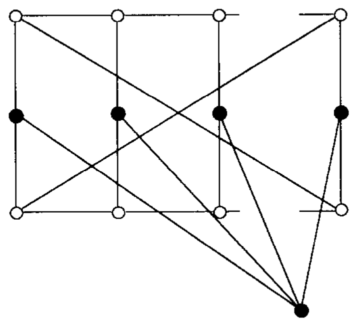


Fig. 5.

It remains now to construct graphs of girth 6 with degree set $\{3, m\}$ on $4m+2$ vertices. The approach we use is not unique, it is used as a convenience since it is easy to check that the girth is indeed 6. For $m \geq 5$ consider m 3-remote edges (a_i, b_i) where a_i and b_i are of degree two. Construct a 'ladder' as shown in Fig. 3. Depending on whether or not m is odd or even we add the edges (x, v) and (u, y) or the edges (x, y) and (u, v). Finally add two new vertices a and b where the neighbours of a and b are $\{a_1, a_2, \ldots, a_m\}$ and $\{b_1, b_2, \ldots, b_m\}$ respectively. The resulting graphs are shown in Fig. 4i and 4ii. The graphs in Fig. 4 have degree sets $\{3, m\}$ and girth 6 as required. Recalling that $f(3, 3; 6) = 14$, the Heawood graph (see [1]), and that $f(3, 4; 6) = 18$ we have the following.

Theorem 1. *The minimal order of a graph of girth* 6 *with degree set* $\{3, m\}$, $m \geq 3$, *is* $f(3, m; 6) = 4m + 2$.

A similar construction using 3-remote vertices instead of edges to form a ladder together with one vertex joined to all of these 3-remote points shows how one may attain the bound of $f(3, m; 5) = 3m + 1$ (see Fig. 5). The case $m = 3$ is the Petersen graph.

The extremal graphs (with respect to number of vertices) are not necessarily unique. For example, if we take a 9-cycle, add three 3-remote vertices of degree three by joining them to every third vertex of this cycle, then we have a 3-regular graph on 12 vertices. Finally join a new vertex to these three extra vertices. The result is a $(3, 4; 5)$-cage with 21 edges as opposed to a $(3, 4{:}\ 5)$-cage with 20 edges in the earlier construction. This begs the question of how many of these generalized cages there are for a given girth and degree set.

3. The $(4, m; 5)$-cages

Theorem 2. *The minimal order of a graph of girth* 5 *with degree set* $\{4, m\}$ *is* $f(4, m; 5) = 4m + 1$, $m \geq 5$.

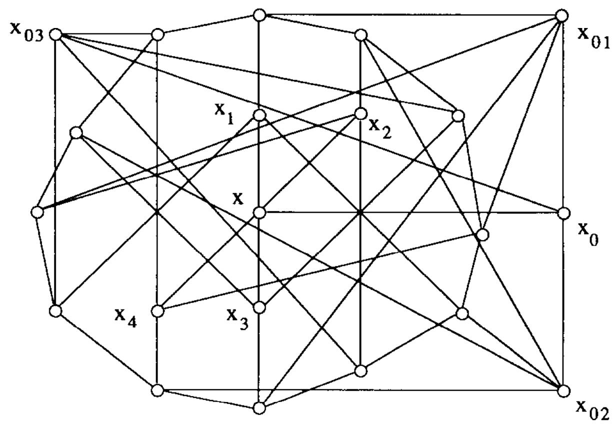


Fig. 6.

Proof. Select a vertex x of degree m. Other than including x, the neighbourhoods of the neighbours of x are disjoint (else the girth is 4), thus $f(4, m; 5) \geq 1 + m + 3m = 4m + 1$. It remains then to show that if $m \geq 5$, then we may construct $(4, m; 5)$-cages of this order.

First define the graph G as follows. Take a $3(m-1)$-cycle with vertices ordered $0, 1, 2, \ldots, 3(m-1)$. Let $X = \{x_1, x_2, \ldots, x_{m-1}\}$ and let x_i be adjacent to vertices $(i-1)$, $(m-1)+(i-1)$ and $2(m-1)+(i-1)$ for $i = 0, 1, \ldots, m-1$. Define vertices x_{01}, x_{02}, x_{03} where x_{0i} is adjacent to those vertices, j, of C with $j = i - 1 \pmod 3$ for $i = 1, 2, 3$. Let x_0 be a new vertex with neighbours x_{01}, x_{02} and x_{03} and finally join a vertex x to $x_0, x_1, x_2, \ldots, x_{m-1}$. Note that in G the vertices x, x_{01}, x_{02}, and x_{03} have degree m while all other vertices are of degree four. It is not difficult to in fact check, for $m \neq 1 \pmod 3$, that G has the properties we desire. In the rest of our proof however we will only appeal to this construction in the cases $m = 5$, 8 and 9. Fig. 6 illustrates the case $m = 5$. For all cases other than $m = 5$, 8 or 9 we adapt the construction in Fig. 5-delete one of the long diagonal edges, subdivide two others and finally add two new edges in order to create a graph H with degree set $\{3, m-1\}$, girth 5 and $f(3, m-1; 5) + 2 = 3m$ vertices. The vertices of H are labelled as shown in Fig. 7 where (other

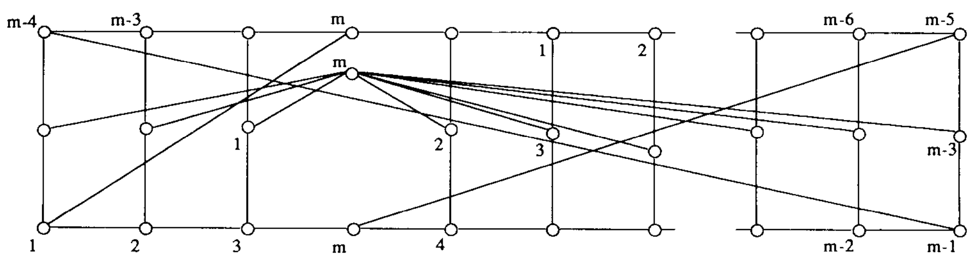


Fig. 7.

than those labelled by m) the labelling is cyclical. H also has the property that the set of vertices labelled i constitute a 3-remote set of vertices for $i = 1, 2, \ldots, m$. Now let G be a graph on $4m + 1$ vertices containing a vertex x_0 of degree m with neighbours $x_1, x_1, \ldots, x_m$. The remaining vertices of G induce the subgraph H of Fig. 7. Also, vertex x_i has those vertices labelled i in H as neighbours for $i = 1, 2, \ldots, m$. Then G has the desired properties and $f(4, m; 5) = 4m + 1$, $m \geqslant 5$. $\square$

When $m = 4$ a different construction leads to the $(4, 5)$-cage, this is the Robertson graph, (see [1]), the unique $(4, 5)$-cage with only 19 vertices.

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